

ANISOTROPIC ORLICZ-SOBOLEV SPACES OF VECTOR VALUED FUNCTIONS AND LAGRANGE EQUATIONS

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ABSTRACT. In this paper we study some properties of anisotropic Orlicz and anisotropic Orlicz-Sobolev spaces of vector valued functions for a special class of G-functions. We introduce a variational setting for a class of Lagrangian Systems. We give conditions which ensure that the principal part of variational functional is finitely defined and continuously differentiable on Orlicz-Sobolev space.

1. INTRODUCTION

In this paper we make some preliminary steps for variational analysis in anisotropic Orlicz-Sobolev spaces of vector valued functions. We consider the Euler-Lagrange equation

$$(1) \quad \frac{d}{dt} L_v(t, u(t), \dot{u}(t)) = L_x(t, u(t), \dot{u}(t)), \quad t \in (a, b)$$

where Lagrangian is of the form $L(t, x, v) = F(t, x, v) + V(t, x)$.

If $F(v) = \frac{1}{2}|v|^2$ then the equation (1) reduces to $\ddot{u}(t) + \nabla V(t, u(t)) = 0$. One can consider more general case $F(v) = \phi(|v|)$, where ϕ is convex and nonnegative. In the above cases F does not depend on v directly but rather on its norm $|v|$ and the growth of F is the same in all directions, i.e. F has isotropic growth. Equation (1) with Lagrangian $L(t, x, v) = \frac{1}{p}|v|^p + V(t, x)$ has been studied by many authors under different conditions. The classical reference is [1]. The isotropic Orlicz-Sobolev space setting was considered in [2].

We are interested in anisotropic case. This means that F depends on all components of v not only on $|v|$ and has different growth in different directions. A simple example of such function is $F(v) = \sum_{i=1}^N |v_i|^{p_i}$ or $F(v) = \sum_{i=1}^N \phi_i(|v_i|)$, where ϕ_i are N-functions. We wish to consider more general situation. We assume that $F: [a, b] \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies

- (F₁) $F \in C^1$,
- (F₂) $|F(t, x, v)| \leq a(|x|)(b(t) + G(v))$,
- (F₃) $|F_x(t, x, v)| \leq a(|x|)(b(t) + G(v))$,
- (F₄) $G^*(F_v(t, x, v)) \leq a(|x|)(c(t) + G^*(\nabla G(v)))$.

where $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b, c \in \mathbf{L}^1(I, \mathbb{R}_+)$ and $G: \mathbb{R}^N \rightarrow \mathbb{R}$ is a G-function. Conditions (F₁)–(F₄) are direct generalization of standard growth conditions from [1] (see also [2]). We show (see Theorem 5.7) that under these conditions the functional $\mathcal{I}: \mathbf{W}^1 \mathbf{L}^G \rightarrow \mathbb{R}$ given by

$$\mathcal{I}(u) = \int_I F(t, u, \dot{u}) dt$$

is continuously differentiable.

We restrict our considerations to a special class of G-functions. Here $G: \mathbb{R}^n \rightarrow [0, \infty)$ is convex, $G(-x) = G(x)$, supercoercive, $G(0) = 0$ and satisfies Δ_2 and ∇_2 conditions. We define the anisotropic Orlicz space to be

$$\mathbf{L}^G(I, \mathbb{R}^N) = \{u: I \rightarrow \mathbb{R}^N: \int_I G(u) dt \leq \infty\}.$$

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The Orlicz space \mathbf{L}^G equipped with the Luxemburg norm

$$\|u\|_{\mathbf{L}^G} = \inf \left\{ \alpha > 0 : \int_I G\left(\frac{u}{\alpha}\right) dt \leq 1 \right\}.$$

is a reflexive Banach space. An important example of Orlicz space is classical Lebesgue \mathbf{L}^p space, defined by $G(x) = \frac{1}{p}|x|^p$. In this case, the Luxemburg norm and the standard \mathbf{L}^p norm are equivalent. Therefore, Orlicz spaces can be viewed as a straightforward generalization of \mathbf{L}^p spaces.

Properties of N-functions and of Orlicz spaces of real-valued functions has been studied in great details in monographs [3, 4, 5] and [6]. The standard references for vector-valued case are [7, 8, 9] and [10, 11] for Banach-space valued functions. In [7, 8] author considers a class of G-functions together with a uniformity conditions which, for example excludes the function $G(x) = \sum |x_i|^{p_i}$ unless $1 < p_1 = \dots = p_N < \infty$. Moreover G is not necessarily assumed to be an even function. As was pointed out in [11], if G is not even then \mathbf{L}^G is no longer a vector space (see also [10, Example 2.1]).

Our strong conditions on G allow us to work in Orlicz spaces without worry about some technical difficulties arising in general case. For example, it is well known that the set $\mathbf{L}^G(I, \mathbb{R}^N)$ is a vector space if and only if G satisfies Δ_2 condition. Otherwise \mathbf{L}^G is only a convex set. Another difficulty is the convergence notion. In Lebesgue spaces $\|u_n - u\|_{\mathbf{L}^p} \rightarrow 0$ means simply $\int |u_n - u|^p \rightarrow 0$. For arbitrary G-function G , convergence in Luxemburg norm is not equivalent to $\int G(u_n - u) dt \rightarrow 0$ unless G satisfies Δ_2 . The Δ_2 condition is also crucial for separability and reflexivity of \mathbf{L}^G .

The main consequence of anisotropic nature of G is the lack of monotonicity of the norm. It is no longer true that $|u| \leq |v|$ implies $\|u\|_{\mathbf{L}^G} \leq \|v\|_{\mathbf{L}^G}$. In anisotropic case, standard dominance condition $|u_n| \leq f$ does not implies convergence in \mathbf{L}^G norm and must be replaced by $G(u_n) \leq f$ (see Theorem 3.14).

Following [10] we show that for every G we consider there exist $p, q \in (1, \infty)$ such that $\mathbf{L}^q \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^p$. If $G(x) = \sum |x_i|^{p_i}$ then \mathbf{L}^G can be identified with the product of \mathbf{L}^{p_i} but in many cases an anisotropic Orlicz Space is not equal to the space $\mathbf{L}^{p_1} \times \mathbf{L}^{p_2} \times \dots \times \mathbf{L}^{p_N}$ (see Example 3.23).

To give a proper variational setting for equation (1) we introduce a notion of an anisotropic Orlicz-Sobolev space $\mathbf{W}^1 \mathbf{L}^G$ of vector-valued functions. It is defined to be

$$\mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^N) = \{u \in \mathbf{L}^G(I, \mathbb{R}^N) : \dot{u} \in \mathbf{L}^G(I, \mathbb{R}^N)\}$$

with the norm

$$\|u\|_{\mathbf{W}^1 \mathbf{L}^G} = \|u\|_{\mathbf{L}^G} + \|\dot{u}\|_{\mathbf{L}^G}$$

To the authors best knowledge there is no reference for the case of anisotropic norm and vector-valued functions of one variable. The references for other cases are [2, 9, 12, 13, 14, 15, 16, 17, 18, 19].

In [9] and [18] the space $H^0(G, \Omega)$, $\Omega \subset \mathbb{R}^n$ is defined as a completion of $C_0^1(\Omega, \mathbb{R}^n)$ under norm $\|u\|_{H^0(G, \Omega)} = \|Du\|_{G, \Omega}$. It is classical result due to Trudinger $H^0(G, \Omega) \hookrightarrow L_A(\Omega)$, where A is some N-function (see also Cianchi [14]).

In [17] and [19] the anisotropic Orlicz-Sobolev space $W^1 L_G$ is defined for G-function $G : \mathbb{R}^{n+1} \rightarrow [0, \infty]$ as a space of weakly differentiable functions $u : \mathbb{R}^n \supset \Omega \rightarrow \mathbb{R}$ such that $(u, D_1 u, D_2 u, \dots, D_n u)$ belongs to the Orlicz space generated by G . A norm for $W^1 L_G$ is given by

$$\|u\|_{1, G, \Omega} = \|(u, Du)\|_{G, \Omega}.$$

In [12] we can find definition of isotropic Orlicz-Sobolev space of real valued functions

$$W_A^1(\Omega) = \{u \in \Omega \rightarrow \mathbb{R} \text{ measurable} : u, |\nabla u| \in L_A\},$$

where L_A is Orlicz Space and A is an N-function.

In [2] the isotropic Orlicz-Sobolev space if vector-valued functions is defined to be a space of absolutely continuous functions $u : [0, T] \rightarrow \mathbb{R}^d$ such that u and \dot{u} belongs to Orlicz space generated by an N-function. Similar treatment can be found in [20].

2. G-FUNCTIONS

Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on \mathbb{R}^N and $|\cdot|$ is the induced norm. We assume that $G : \mathbb{R}^N \rightarrow [0, \infty)$ satisfies the following conditions:

- (G₁) $G(0) = 0$,
- (G₂) G is convex,
- (G₃) G is even,
- (G₄) G is supercoercive:

$$\lim_{|x| \rightarrow \infty} \frac{G(x)}{|x|} = \infty,$$

(G₅) G satisfies the Δ_2 condition:

$$(\Delta_2) \quad \exists_{K_1 \geq 2} \exists_{M_1 > 0} \forall_{|x| \geq M_1} G(2x) \leq K_1 G(x),$$

(G₆) G satisfies the ∇_2 condition:

$$(\nabla_2) \quad \exists_{K_2 \geq 1} \exists_{M_2 > 0} \forall_{|x| \geq M_2} G(x) \leq \frac{1}{2K_2} G(K_2 x).$$

A function G is a G-function in the sense of Trudinger [9]. In general, G-function can be unbounded on bounded sets and need not satisfy conditions (G₄)–(G₆) but only $\lim_{x \rightarrow \infty} G(x) = \infty$. A G-function of one variable is called N-function. Some typical examples of G are

- (1) $G(x) = \frac{1}{p}|x|^p$, $1 < p < \infty$
- (2) $G(x) = \sum_{i=1}^N G_{p_i}(x_i)$, $1 < p_i < \infty$
- (3) $G(x) = (x_1 - x_2)^2 + x_2^4$

A function G can be equal to zero in some neighborhood of 0. So that a function

$$G(x) = \begin{cases} 0 & |x| \leq 1 \\ |x|^2 - 1 & |x| > 1 \end{cases}$$

is also admissible. Conditions Δ_2 and ∇_2 implies that G is of polynomial growth (see Lemma 2.4 below and [3]). A function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ $f(x) = e^{|x|} - |x| - 1$ does not satisfy Δ_2 .

Since G is convex and finite on \mathbb{R}^n , G is locally Lipschitz and therefore continuous. Note that for every $x \in \mathbb{R}^N$

$$\begin{aligned} G(\alpha x) &\leq \alpha G(x), \text{ if } 0 \leq \alpha \leq 1, \\ \alpha G(x) &\leq G(\alpha x), \text{ if } 1 \leq \alpha. \end{aligned}$$

We obtain immediately that G is non-decreasing along any half-line through the origin i.e. for every $x \in \mathbb{R}^N$

$$(2) \quad 0 < \alpha \leq \beta \implies G(\alpha x) \leq G(\beta x).$$

Our assumptions on G imply that for every $x_0 \in \mathbb{R}^N$ there exists $a \in \mathbb{R}^N$ and $b \in \mathbb{R}$ such that for all $x \in \mathbb{R}^N$

$$\langle a, x_0 \rangle + b = G(x_0) \text{ and } \langle a, x \rangle + b \leq G(x).$$

From this, we can easily obtain the Jensen integral inequality. Let $I \subset \mathbb{R}$ be a finite interval and let $u \in \mathbf{L}^1(I, \mathbb{R}^N)$. Then

$$G\left(\frac{1}{\mu(I)} \int_I u \, dt\right) \leq \frac{1}{\mu(I)} \int_I G(u) \, dt.$$

We will often make use of the following simple observation.

Proposition 2.1. For all $\alpha \in \mathbb{R}$ there exists $K_1(\alpha) > 0$ such that

$$G(\alpha x) \leq K_1(\alpha) G(x)$$

for all $|x| \geq M_1$.

In fact, the above proposition provides a characterization of Δ_2 (see [7, 11]). It follows that for every $\alpha \in \mathbb{R}$ there exists $C_\alpha > 0$ such that for $x \in \mathbb{R}^N$

$$G(\alpha x) \leq C_\alpha + K_1(\alpha) G(x).$$

We recall a notion of Fenchel conjugate. Define $G^* : \mathbb{R}^N \rightarrow [0, \infty)$ by

$$G^*(y) := \sup_{x \in \mathbb{R}^N} \{\langle x, y \rangle - G(x)\}.$$

A function G^* is called Fenchel conjugate of G . As an immediate consequence of definition we have the so called Fenchel inequality:

$$\forall_{x, y \in \mathbb{R}^N} \langle x, y \rangle \leq G(x) + G^*(y).$$

Consider arbitrary $f : \mathbb{R}^N \rightarrow [0, \infty)$. It is obvious that the conjugate function f^* is always convex. But in general f^* need not be continuous, finite or coercive, even if f is. From the other hand, it is well known that if f is convex and l.s.c. then $f^* \neq \infty$ and $(f^*)^* = f$.

Example 2.2. (1) If

$$g(x) = \begin{cases} 0 & |x| \leq 1 \\ \infty & |x| > 1 \end{cases}$$

then $g^*(x) = |x|$. Note that g and g^* are G-functions but do not satisfy our assumptions.

- (2) If $G(x) = \frac{1}{p}|x|^p$, then $G^*(x) = \frac{1}{q}|x|^q$, $\frac{1}{p} + \frac{1}{q} = 1$.
- (3) If $G(x) = \sum_{i=1}^N G_i(x_i)$ then $G^*(x) = \sum_{i=1}^N G_i^*(x_i)$.
- (4) If $G(x, y) = (x - y)^2 + y^4$, then

$$G^*(x, y) = \frac{1}{4}x^2 + \frac{3}{4}(x + y) \left(\frac{x + y}{4} \right)^{\frac{1}{3}}.$$

More information on general theory of conjugate functions can be found in standard books on convex analysis, see for instance [21, 22].

If a function $G : \mathbb{R}^n \rightarrow [0, \infty)$ satisfies conditions (G_1) – (G_6) then the same is true for its conjugate G^* . This is main reason we want to restrict class of considered functions.

Theorem 2.3. If G satisfies conditions (G_1) – (G_6) then G^* also satisfies (G_1) – (G_6) and $(G^*)^* = G$.

Proof. It is evident that G^* satisfies (G_1) , (G_2) and (G_3) . It is well known that under our conditions, G^* is finite (proposition 1.3.8, [21]), G^* is supercoercive (proposition 1.3.9, [21]) and G^* satisfies (G_5) and (G_6) (remark 2.3, [10]). Corollary [21, cor. 1.3.6] gives $(G^*)^* = G$. \square

In order to compare growth rate of G-functions we define two relations. Let G_1 and G_2 be G-functions. Define

$$G_1 \prec G_2 \iff \exists_{M \geq 0} \exists_{K > 0} \forall_{|x| \geq M} G_1(x) \leq G_2(Kx)$$

and

$$G_1 \prec\prec G_2 \iff \forall_{\alpha > 0} \lim_{|x| \rightarrow \infty} \frac{G_2(\alpha x)}{G_1(x)} = \infty.$$

For conjugate functions we have (see [3, thm. 3.1])

$$G_1 \prec G_2 \Rightarrow G_2^* \prec G_1^*.$$

Obviously $G_1 \prec\prec G_2$ implies $G_1 \prec G_2$. Assumption (G_4) implies $|x| \prec\prec G$. It is true that $|x| \prec G$ holds under weaker assumption: $G(x) \rightarrow \infty$. Note that, if $p > 1$ then $|x| \prec\prec |x|^p$. Hence, if $|x|^p \prec G$ then $|x| \prec\prec G$. Since G satisfies (G_5) and (G_6) we have the following bounds for the growth of G .

Lemma 2.4 (cf. [10, Lemma 2.4]). There exists $p, q \in (1, \infty)$ such that

$$|x|^p \prec G \prec |x|^q.$$

Proof. Set $C = \overline{G}(M_1)$. By induction, if $|x| \leq 2^n M_1$ then $G(x) \leq K_1^n C$. For $|x| \geq M_1$ choose n such that $2^{n-1} M_1 \leq |x| \leq 2^n M_1$. Then $n - 1 \leq \log_2(|x|/M_1)$ and $G(x) \leq C K_1^{1 + \log_2(|x|/M_1)}$. Therefore, for $|x| \geq M_1$,

$$G(x) \leq C K_1 M_1^{-q} |x|^q, \quad q = \log_2(K_1).$$

This proves that $G \prec |x|^q$. Choose $r > 0$ such that if $x \in G^{-1}(\overline{G}(M_1))$ then $|x| \leq r$. Set $M = rM_1$. Again, by induction, for $|x| \geq K_2^k M$ we have $(2K_2)^k C \leq G(x)$. This implies

$$G(x) \geq C(2K_2)^{-q}|x|^q, \quad p = 1 + \frac{1}{\log_2(K_2)}$$

whenever $|x| \geq MK_2$. Hence $|x|^p \prec G$. □

Immediately from the above we get $|x|^{\frac{q}{q-1}} \prec G^* \prec |x|^{\frac{p}{p-1}}$.

3. ORLICZ SPACES

Let $I \subset \mathbb{R}$ be a finite interval. The Orlicz space $\mathbf{L}^G = \mathbf{L}^G(I, \mathbb{R}^n)$ is defined to be

$$\mathbf{L}^G(I, \mathbb{R}^n) = \left\{ u: I \rightarrow \mathbb{R}^n: u \text{ - measurable } \int_I G(u) dt < \infty \right\}.$$

As usual, we identify functions equal a.e. For an arbitrary G-function $f: \mathbb{R}^n \rightarrow [0, \infty)$ which does not satisfies Δ_2 the set \mathbf{L}^f is not a linear space but only a convex set. In fact, it is well known that the set \mathbf{L}^f is linear space if and only if a G-function f satisfies Δ_2 condition.

For $u \in \mathbf{L}^G$ define:

$$\|u\|_{\mathbf{L}^G} = \inf \left\{ \alpha > 0: \int_I G\left(\frac{u}{\alpha}\right) dt \leq 1 \right\}.$$

The function $\|\cdot\|_{\mathbf{L}^G}$ is called the Luxemburg norm. It is easy to see that

$$\int_I G\left(\frac{u}{\|u\|_{\mathbf{L}^G}}\right) dt = 1,$$

since G satisfies Δ_2 . Moreover

$$\int_I G\left(\frac{u}{k}\right) dt \leq 1 \iff \|u\|_{\mathbf{L}^G} \leq k.$$

Remark 3.1. All properties of \mathbf{L}^G remains true for \mathbf{L}^{G^*} , since G and G^* belongs to the same class of functions.

Theorem 3.2. If $G: \mathbb{R}^n \rightarrow [0, \infty)$ satisfies (G_1) – (G_6) , then $(\mathbf{L}^G(I, \mathbb{R}^n), \|\cdot\|_{\mathbf{L}^G})$ is a normed linear space.

Proof. We first prove that \mathbf{L}^G is a linear space. Since G is continuous and satisfies Δ_2 , we get

$$\int_I G(\alpha u) dt = \int_{I_1} G(\alpha u) dt + \int_{I \setminus I_1} G(\alpha u) dt \leq \mu(I_1) C_\alpha + K_1(\alpha) \int_I G(u) dt < \infty$$

where $I_1 = \{t \in I: |u(t)| \leq M_1\}$. Hence, if $u \in \mathbf{L}^G$ then $\alpha u \in \mathbf{L}^G$ for all $\alpha \in \mathbb{R}$. For every $u, v \in \mathbf{L}^G$ and $\alpha, \beta \in \mathbb{R}$, by (G_2) and Proposition 2.1, we have

$$\int_I G(\alpha u + \beta v) dt \leq \frac{1}{2} \int_I G(2\alpha u) dt + \frac{1}{2} \int_I G(2\beta v) dt < \infty.$$

Hence $\alpha u + \beta v \in \mathbf{L}^G$.

Now we show that $\|\cdot\|_{\mathbf{L}^G}$ is a norm on \mathbf{L}^G . It is evident that if $u = 0$ then $\|u\|_{\mathbf{L}^G} = 0$. Suppose $u \neq 0$. There exists $I_1 \subset I$ with positive measure and $\varepsilon > 0$ such that for all $t \in I_1$, $|u(t)| \geq \varepsilon$. For every $t \in I_1$ there exists $\alpha_t \geq 1$ and $y_t \in \mathbb{R}^n$, $|y_t| = \varepsilon$ such that $u(t) = \alpha_t y_t$. For all $k > 0$ we have by (2) that $G(\alpha_t y_t/k) \geq G(y_t/k)$. Hence

$$\int_I G\left(\frac{u}{k}\right) dt \geq \int_{I_1} G\left(\frac{u}{k}\right) dt = \int_{I_1} G\left(\frac{\alpha_t y_t}{k}\right) dt \geq \int_{I_1} G\left(\frac{y_t}{k}\right) dt \geq \mu(I_1) \underline{G}(\varepsilon/k),$$

where $\underline{G}(\varepsilon/k) = \inf\{G(y): |y| = \frac{\varepsilon}{k}\}$. Since $\underline{G}(\varepsilon/k) \nearrow \infty$ as $k \searrow 0$, there exists $k_0 > 0$ such that for all $k \leq k_0$

$$\int_I G\left(\frac{u}{k}\right) dt > 1$$

and

$$\|u\|_{\mathbf{L}^G} = \inf \left\{ k > 0 : \int_I G\left(\frac{u}{k}\right) dt \leq 1 \right\} \geq k_0 > 0.$$

Finally, $\|u\|_{\mathbf{L}^G} = 0 \iff u = 0$. Let $u \in L^G$ and $\alpha \in \mathbb{R}$. For $\alpha \in \mathbb{R}$:

$$\|\alpha u\|_{\mathbf{L}^G} = \inf \left\{ k > 0 : \int_I G\left(\frac{\alpha u}{k}\right) \leq 1 \right\} = |\alpha| \inf \left\{ k/|\alpha| > 0 : \int_I G\left(\frac{u}{k/|\alpha|}\right) \leq 1 \right\} = |\alpha| \|u\|_{\mathbf{L}^G}.$$

If $\|u\|_{\mathbf{L}^G} = 0$ or $\|v\|_{\mathbf{L}^G} = 0$, then it is obvious that $\|u + v\|_{\mathbf{L}^G} \leq \|u\|_{\mathbf{L}^G} + \|v\|_{\mathbf{L}^G}$. Set $\alpha = \|u\|_{\mathbf{L}^G} > 0$, $\beta = \|v\|_{\mathbf{L}^G} > 0$. Then $\int_I G\left(\frac{u}{\alpha}\right) = 1$ and $\int_I G\left(\frac{v}{\beta}\right) = 1$. Thus

$$\int_I G\left(\frac{u+v}{\alpha+\beta}\right) dt \leq \frac{\alpha}{\alpha+\beta} \int_I G\left(\frac{u}{\alpha}\right) dt + \frac{\beta}{\alpha+\beta} \int_I G\left(\frac{v}{\beta}\right) dt = 1.$$

As a consequence

$$\int_I G\left(\frac{u+v}{\|u\|_{\mathbf{L}^G} + \|v\|_{\mathbf{L}^G}}\right) dt \leq 1 \implies \|u+v\|_{\mathbf{L}^G} \leq \|u\|_{\mathbf{L}^G} + \|v\|_{\mathbf{L}^G}.$$

□

An important example of Orlicz space is a classical Lebesgue space $(\mathbf{L}^p, \|\cdot\|_{\mathbf{L}^p})$, $p \in (1, \infty)$ defined by $G(x) = \frac{1}{p}|x|^p$. It is easy to check that in this case $\mathbf{L}^G = \mathbf{L}^p$ and the Luxemburg norm and standard \mathbf{L}^p norm are equivalent. Two important examples of Lebesgue spaces are not covered in our setting, namely \mathbf{L}^1 and \mathbf{L}^∞ . The space \mathbf{L}^1 is generated by $f(x) = |x|$ and the space \mathbf{L}^∞ generated by f^* . We exclude these two spaces because we want to have only reflexive spaces in the class of Orlicz spaces we consider.

It was pointed out by Schappacher [11, example 3.1] that if f is not bounded on bounded sets (i.e. we allow $f(x) = +\infty$ for some $x \in \mathbb{R}^n$) then \mathbf{L}^f need not be a linear space, even if f satisfies Δ_2 condition. To see this, consider

$$f(x) = \begin{cases} \frac{1}{1-|x|} - 1 & |x| < 1 \\ \infty & |x| \geq 1 \end{cases} \text{ and } u(t) = t/2$$

See [3, 11] for more details.

Theorem 3.3 (Hölder inequality). For every $u \in \mathbf{L}^G$ and $v \in \mathbf{L}^{G^*}$

$$\int_I \langle u, v \rangle dt \leq 2 \|u\|_{\mathbf{L}^G} \|v\|_{\mathbf{L}^{G^*}}$$

Proof. Using Fenchel inequality we obtain

$$\left\langle \frac{u}{\|u\|_{\mathbf{L}^G}}, \frac{v}{\|v\|_{\mathbf{L}^{G^*}}} \right\rangle \leq G\left(\frac{u}{\|u\|_{\mathbf{L}^G}}\right) + G^*\left(\frac{v}{\|v\|_{\mathbf{L}^{G^*}}}\right).$$

Hence

$$\int_I \left\langle \frac{u}{\|u\|_{\mathbf{L}^G}}, \frac{v}{\|v\|_{\mathbf{L}^{G^*}}} \right\rangle dt \leq \int_I G\left(\frac{u}{\|u\|_{\mathbf{L}^G}}\right) dt + \int_I G^*\left(\frac{v}{\|v\|_{\mathbf{L}^{G^*}}}\right) dt \leq 2.$$

□

We finish this section by completeness of Orlicz space.

Theorem 3.4 (cf. [3], [11, theorem 6.1]). The space $(\mathbf{L}^G, \|\cdot\|_{\mathbf{L}^G})$ is complete.

Proof. Let $\{u_n\}$ be a Cauchy sequence in \mathbf{L}^G . Fix $\delta, \epsilon > 0$ and choose $\alpha > 0$ such that $G(\alpha x) > 2/\delta$ if $|x| \geq \epsilon$. Let n_0 be large enough so that $\|u_n - u_m\|_{\mathbf{L}^G} \leq \alpha^{-1}$, i.e

$$\int_I G(\alpha(u_n - u_m)) dt \leq 1.$$

Put $E = \{t: G(\alpha(u_n(t) - u_m(t))) > \delta/2\}$. Then

$$\frac{2}{\delta} \mu(E) \leq \int_I G(\alpha(u_n - u_m)) dt \leq 1$$

that is $\mu(E) < \frac{\delta}{2}$. It follows that

$$\mu(\{t: |u_n(t) - u_m(t)| \geq \varepsilon\}) \leq \frac{\delta}{2}$$

Thus $\{u_n\}$ is a Cauchy sequence in measure. This follows that there is a subsequence $\{u_{n_k}\}$ convergent a.e. to some measurable function u .

Fix $\varepsilon > 0$ and choose K such that for $k, l > K$, $\|u_{n_k} - u_{n_l}\|_{\mathbf{L}^G} \leq \varepsilon$. Then

$$\int_I G\left(\frac{u_{n_k} - u_{n_l}}{\varepsilon}\right) dt \leq \int_I G\left(\frac{u_{n_k} - u_{n_l}}{\|u_{n_k} - u_{n_l}\|_{\mathbf{L}^G}}\right) dt = 1.$$

Letting $n_l \rightarrow \infty$ we obtain by Fatou Lemma,

$$\int_I G\left(\frac{u_{n_k} - u}{\varepsilon}\right) dt \leq 1.$$

Hence $u_{n_k} - u \in \mathbf{L}^G$ and consequently $u \in \mathbf{L}^G$. Since $\varepsilon > 0$ is arbitrary, $\|u_{n_k} - u\|_{\mathbf{L}^G} \rightarrow 0$ and $\|u_n - u\|_{\mathbf{L}^G} \rightarrow 0$. \square

3.1. Convergence. Now we investigate relations between Luxemburg norm and the integral

$$R_G(u) := \int_I G(u) dt.$$

A functional R_G is called modular. Theory of modulars is well known and is developed in more general setting than ours. More information can be found in [23, 5].

For Lebesgue spaces a notions of modular and norm are indistinguishable because modular $\int_I |u|^p dt$ is equal to $\|u\|_{\mathbf{L}^p}^p$. But in Orlicz spaces relation between R_G and $\|\cdot\|_{\mathbf{L}^G}$ is more complex.

There is remarkable difference between isotropic and anisotropic spaces. It is clear that if $u, v \in \mathbf{L}^p$ (or more generally in isotropic Orlicz space) then $|u(t)| \leq |v(t)|$ a.e. implies $\|u\|_{\mathbf{L}^p} \leq \|v\|_{\mathbf{L}^p}$. In anisotropic case it is no longer true, even if $G(u(t)) < G(v(t))$. Next two examples illustrates this point.

Example 3.5. Let $G(x, y) = (x - y)^2 + y^4$, $I = [0, 1]$, $u(t) = (2, 0)$ and $v(t) = (2, 3/2)$. Then $|u(t)| < |v(t)|$, $G(u(t)) < G(v(t))$ and $R_G(u) \leq R_G(v)$, but $2 = \|u\|_{\mathbf{L}^G} > \|v\|_{\mathbf{L}^G} \simeq 1.6$.

Example 3.6. Let $G(x, y) = x^2 + y^4$, $u(t) = (1, 0)$ and $v(t) = \frac{11}{10}(\cos t, \sqrt{\sin t})$. In $\mathbf{L}^G([0, \pi], \mathbb{R}^2)$ we have

$$\sqrt{\pi} = \|u\|_{\mathbf{L}^G} > \|v\|_{\mathbf{L}^G} \simeq 1.7$$

but $|u(t)| < |v(t)|$, $G(u(t)) < G(v(t))$ for all $t \in [0, \pi]$ and $R_G(u) < R_G(v)$.

Definition 3.7. We say that a subset $K \subset \mathbf{L}^G$ is modular bounded if there exists $C > 0$ such that

$$R_G(u) \leq C, \text{ for all } u \in K.$$

Modular boundedness is sometimes called mean boundedness. It is evident that $R_G(u) \leq \|u\|_{\mathbf{L}^G}$ if $\|u\|_{\mathbf{L}^G} \leq 1$ and $R_G(u) > \|u\|_{\mathbf{L}^G}$ if $\|u\|_{\mathbf{L}^G} > 1$.

Lemma 3.8. Let $u \in \mathbf{L}^G$.

- (1) If $R_G(u) \leq C$ then $\|u\|_{\mathbf{L}^G} \leq \max\{C, 1\}$.
- (2) If $\|u\|_{\mathbf{L}^G} \leq C$ then $R_G(u) \leq \mu(I)\tilde{C} + K_1(C)$ for some $\tilde{C} > 0$.

Moreover, a set $K \subset \mathbf{L}^G$ is modular bounded if and only if is norm bounded.

Proof. Assume that $R_G(u) \leq C$. If $C \leq 1$ then $\|u\|_{\mathbf{L}^G} \leq 1$. If $C > 1$ then

$$\int_I G\left(\frac{u}{C}\right) dt \leq \frac{1}{C} \int_I G(u) dt \leq 1.$$

This implies $\|u\|_{\mathbf{L}^G} \leq \max\{C, 1\}$. For the second statement, assume $\|u\|_{\mathbf{L}^G} \leq C$. Then

$$R_G(u) = \int_{I_1} G(u) dt + \int_{I \setminus I_1} G\left(C \frac{u}{C}\right) dt \leq \mu(I_1)\tilde{C} + K_1(C) \int_I G\left(\frac{u}{C}\right) dt,$$

where $I_1 = \{t \in I : |u(t)| \leq M_1 C\}$ and $\tilde{C} > 0$. To finish the proof observe that

$$\int_I G\left(\frac{u}{C}\right) dt \leq \int_I G\left(\frac{u}{\|u\|_{\mathbf{L}^G}}\right) dt = 1.$$

□

Definition 3.9. We say that a sequence of functions $u_k \in \mathbf{L}^G$ is modular convergent to $u \in \mathbf{L}^G$ if $R_G(u_k - u) \rightarrow 0$ as $k \rightarrow \infty$.

Modular convergence is sometimes called mean convergence. Norm convergence always implies modular convergence. Let $\|u_k\|_{\mathbf{L}^G} \rightarrow 0$ as $k \rightarrow \infty$. We can assume that $\forall_k \|u_k\|_{\mathbf{L}^G} \leq 1$, then

$$\frac{1}{\|u_k\|_{\mathbf{L}^G}} R_G(u_k) \leq R_G\left(\frac{u_k}{\|u_k\|_{\mathbf{L}^G}}\right) = 1.$$

Hence $0 \leq R_G(u_k) \leq \|u_k\|_{\mathbf{L}^G}$. In general, converse is not true unless G satisfies Δ_2 condition. (see [3, 11]).

Theorem 3.10. Norm convergence is equivalent to modular convergence.

Proof. We need only to prove that modular convergence implies norm convergence. Fix $\varepsilon > 0$ and assume that $\{u_k\}$ is modular convergent to 0. Define

$$I_{1,k} = \{t \in I : |u_k(t)| \leq M_1\}$$

Since G satisfies Δ_2 , for all $k > 0$ we have

$$\int_I G(u_k/\varepsilon) dt \leq \mu(I_{1,k}) C_{M_1} + K_1(1/\varepsilon) \int_{I \setminus I_{1,k}} G(u_k) dt \leq \mu(I) C_{M_1} + K_1(1/\varepsilon) \int_I G(u_k) dt.$$

For sufficiently large k we have

$$\int_I G(u_k) dt \leq \frac{1}{K_1(1/\varepsilon)}$$

and

$$\int_I G(u_k/\varepsilon) dt \leq \mu(I) C_{M_1} + 1 = C.$$

Finally, Lemma 3.8 shows that $\|u_k\|_{\mathbf{L}^G} \leq C\varepsilon$ and hence $\|u_k\|_{\mathbf{L}^G} \rightarrow 0$. □

It is standard result due to Riesz that for $f_n, f \in \mathbf{L}^p$

$$f_n \rightarrow f \text{ a.e.} \implies \|f_n\|_{\mathbf{L}^p} \rightarrow \|f\|_{\mathbf{L}^p} \iff \|f_n - f\|_{\mathbf{L}^p} \rightarrow 0.$$

Following lemmas establish Orlicz space version of this fact.

Lemma 3.11. For every $k > 1$ and $0 < \varepsilon < \frac{1}{k}$ and $x, y \in \mathbb{R}^n$

$$|G(x+y) - G(x)| \leq \varepsilon |G(kx) - kG(x)| + 2G(C_\varepsilon y)$$

where $C_\varepsilon = \frac{1}{\varepsilon(k-1)}$

Proof. The proof is due to Brezis and Lieb [24] (see also [25]). We repeat the proof. Let $\alpha = 1 - k\varepsilon$, $\beta = \varepsilon$, $\gamma = \varepsilon(k-1)$. Then $\alpha + \beta + \gamma = 1$ and $x + y = \alpha x + \beta(kx) + \gamma(C_\varepsilon y)$. By convexity

$$G(x+y) \leq \alpha G(x) + \beta G(kx) + \gamma G(C_\varepsilon y).$$

This implies that

$$G(x+y) - G(x) \leq \varepsilon(G(kx) - kG(x)) + G(C_\varepsilon y).$$

For the reverse inequality let

$$\alpha = \frac{1}{1+k\varepsilon}, \quad \beta = \frac{\varepsilon}{1+k\varepsilon}, \quad \gamma = \frac{\varepsilon(k-1)}{1+k\varepsilon}.$$

Then $x = \alpha(x+y) + \beta(kx) + \gamma(-C_\varepsilon y)$ and

$$G(x) - G(x+y) \leq \varepsilon(G(kx) - kG(x)) + \varepsilon(k-1)G(C_\varepsilon y).$$

□

Lemma 3.12. If $u_n \rightarrow u$ in \mathbf{L}^G then $R_G(u_n) \rightarrow R_G(u)$.

Proof. In Lemma 3.11 set $x + y = u_n$, $x = u$, $k = 2$. Then $\varepsilon < 1/2$, $C_\varepsilon = \frac{1}{\varepsilon}$ and

$$|G(u_n) - G(u)| \leq \varepsilon |G(2u) - 2G(u)| + 2G\left(\frac{u_n - u}{\varepsilon}\right).$$

Since $u_n \rightarrow u$ in \mathbf{L}^G , there exists n_0 such that for $n > n_0$ we have $\|u_n - u\|_{\mathbf{L}^G} < \varepsilon^2 \leq \varepsilon < 1$. Thus

$$\int_I G\left(\frac{u_n - u}{\varepsilon}\right) dt \leq \frac{1}{\varepsilon} \|u_n - u\|_{\mathbf{L}^G} < \varepsilon.$$

From this and inequality above we obtain

$$|R_G(u_n) - R_G(u)| \leq \varepsilon \int_I |G(2u) - 2G(u)| dt + 2\varepsilon.$$

Letting $\varepsilon \rightarrow 0$ we have $R_G(u_n) \rightarrow R_G(u)$. □

Norm convergence $u_n \rightarrow u$ in \mathbf{L}^p implies that there exists a subsequence such that $u_{n_k} \rightarrow u$ a.e. and $|u_{n_k}| \leq |h| \in \mathbf{L}^p$. According to the above lemma, if $u_n \rightarrow u$ in \mathbf{L}^G then:

(1) Since $\mathbf{L}^G \hookrightarrow \mathbf{L}^1$ (see Lemma 3.20 below), we can extract a subsequence u_{n_k} such that

$$u_{n_k} \rightarrow u \text{ a.e and } |u_{n_k}| \leq h \in \mathbf{L}^1(I, \mathbb{R}).$$

(2) Since $R_G(u_n - u) \rightarrow 0$, $G(u_n - u) \rightarrow 0$ in \mathbf{L}^1 . Thus we can extract a subsequence $\{u_{n_k}\}$ such that

$$G(u_{n_k} - u) \rightarrow 0 \text{ a.e and } G(u_{n_k} - u) \leq h \in \mathbf{L}^1(I, \mathbb{R}).$$

(3) Since $R_G(u_n) \rightarrow R_G(u)$, $G(u_n) \rightarrow G(u)$ in \mathbf{L}^1 . Hence there exists a subsequence $\{u_{n_k}\}$ such that

$$G(u_{n_k}) \rightarrow G(u) \text{ a.e and } G(u_{n_k}) \leq h \in \mathbf{L}^1(I, \mathbb{R}).$$

Lemma 3.13. Let $\{u_n\} \subset \mathbf{L}^G$ and $u \in \mathbf{L}^G$. Suppose that

- (1) $u_n \rightarrow u$ a.e.
- (2) $R_G(u_n) \rightarrow R_G(u)$.

Then $u_n \rightarrow u$ in \mathbf{L}^G .

Proof. This lemma was proved in [4, p. 83] for N-functions. Since G is convex, we get $\frac{1}{2}(G(u_n(t)) + G(u(t))) - G(\frac{u_n(t) - u(t)}{2}) \geq 0$. Continuity of G and $u_n \rightarrow u$ a.e. implies

$$\frac{1}{2}(G(u_n(t)) + G(u(t))) - G\left(\frac{u_n(t) - u(t)}{2}\right) \rightarrow G(u) \text{ a.e.}$$

So that by the Fatou Lemma, we have

$$\begin{aligned} \int_I G(u) dt &\leq \liminf_{n \rightarrow \infty} \int_I \frac{1}{2}(G(u_n) + G(u)) dt - \limsup_{n \rightarrow \infty} \int_I G\left(\frac{u_n - u}{2}\right) dt \leq \\ &\leq \lim_{n \rightarrow \infty} \int_I \frac{1}{2}(G(u_n) + G(u)) dt - \limsup_{n \rightarrow \infty} \int_I G\left(\frac{u_n - u}{2}\right) dt = \\ &= \int_I G(u) dt - \limsup_{n \rightarrow \infty} \int_I G\left(\frac{u_n - u}{2}\right) dt. \end{aligned}$$

This implies that

$$\int_I G\left(\frac{u_k(t) - u(t)}{2}\right) dt \rightarrow 0$$

and $\|u_k - u\|_{\mathbf{L}^G} \rightarrow 0$ by Theorem 3.10. □

As a consequence we obtain dominated convergence theorem for anisotropic Orlicz spaces:

Theorem 3.14. Suppose that $\{u_n\} \subset \mathbf{L}^G$ and

- (1) $u_n \rightarrow u$ a.e.

(2) there exists $h \in \mathbf{L}^1$ such that $G(u_n) \leq h$ a.e.

Then $u \in \mathbf{L}^G$ and $u_n \rightarrow u$ in \mathbf{L}^G .

Proof. Since G is continuous and $u_n \rightarrow u$ a.e., $G(u_n) \rightarrow G(u)$ a.e. It follows that $G(u) \leq h$ a.e. Thus $G(u) \in \mathbf{L}^1$ and hence $u \in \mathbf{L}^G$. Since $0 \geq h \pm G(u_n)$ and $h \pm G(u_n) \rightarrow h \pm G(u)$ a.e., application of the Fatou Theorem yields

$$\int_I h \, dt \pm \int_I G(u_n) \, dt \leq \liminf \int_I h \, dt \pm G(u_n) \, dt.$$

Therefore,

$$\begin{aligned} \int_I h \, dt + \int_I G(u) \, dt &\leq \int_I h \, dt + \liminf \int_I G(u_n) \, dt \\ \int_I h \, dt - \int_I G(u) \, dt &\leq \int_I h \, dt - \limsup \int_I G(u_n) \, dt \end{aligned}$$

and hence

$$\limsup \int_I G(u_n) \, dt \leq \int_I G(u) \, dt \leq \liminf \int_I G(u_n) \, dt$$

and $R_G(u_n) \rightarrow R_G(u)$. By the Lemma 3.13, $u_n \rightarrow u$ in \mathbf{L}^G . \square

In the above theorem, assumption $G(u_n) \leq h$ can be replaced by $G(u_n) \leq G(h)$, $h \in \mathbf{L}^G$. Consider a sequence $\{u_n\} \subset \mathbf{L}^G$ convergent pointwise to measurable function u . Under standard dominance condition (i.e. $|u_n| \leq |g|$, $g \in \mathbf{L}^G$) it is not true in general that $u_n \rightarrow u \in \mathbf{L}^G$.

Example 3.15. Let $G(x, y) = x^2 + y^4$, $I = (0, 1)$, $u(t) = (0, t^{-1/4})$ and $h(t) = (t^{-3/8}, 0)$. Define

$$u_n(t) = \begin{cases} u(t) & |u(t)| \leq n \\ 0 & |u(t)| > n \end{cases}$$

Then $u_n \rightarrow u$ a.e., $u_n, h \in \mathbf{L}^G$ and $|u_n| \leq |h|$ for every t . But $G(u(t)) = t^{-1} \notin \mathbf{L}^1(I, \mathbb{R})$. Hence $u \notin \mathbf{L}^G$.

Remark 3.16. Modular R_G is called monotone modular if $|x| \leq |y|$ implies $R_G(x) \leq R_G(y)$. If R_G is monotone modular then $u_k \rightarrow u$ a.e and $|u_k| \leq |g|$, $g \in \mathbf{L}^G$ implies $u \in \mathbf{L}^G$ and $\|u_k - u\|_{\mathbf{L}^G} \rightarrow 0$. We refer the reader to [25] for more details.

3.2. Separability. For every $u \in \mathbf{L}^G$ there exists a sequence of bounded functions $\{u_n\} \subset \mathbf{L}^G$ such that $u_n \rightarrow u$ in \mathbf{L}^G . For example, one can define

$$u_n(t) = \begin{cases} u(t) & |u(t)| \leq n \\ 0 & |u(t)| > n \end{cases}$$

In this case $u_n \rightarrow u$ a.e and $G(u_n(t) - u(t)) \leq G(u(t))$. Therefore, by Theorem 3.14 we get $u_n \rightarrow u$ in \mathbf{L}^G .

Theorem 3.17 (cf. [3, p. 81]). The space \mathbf{L}^G is separable.

Proof. Fix $\varepsilon > 0$. Suppose that $u \in \mathbf{L}^G$ is bounded and $|u(t)| \leq a$. Set $C = \sup\{G(x/\varepsilon) : |x| \leq 2a\}$.

By the Luzin theorem we can find a compact subset $I_1 \subset I$ and a continuous function $u_1 : I \rightarrow \mathbb{R}^N$ such that $\mu(I \setminus I_1) \leq 1/C$, $u(t) = u_1(t)$ for all $t \in I_1$ and $|u_1(t)| \leq a$. Now we get

$$\int_I G\left(\frac{u - u_1}{\varepsilon}\right) dt = \int_{I \setminus I_1} G\left(\frac{u - u_1}{\varepsilon}\right) dt \leq \mu(I \setminus I_1)C \leq 1.$$

Hence $\|u - u_1\|_{\mathbf{L}^G} \leq \varepsilon$. For arbitrary $v \in \mathbf{L}^G$ we can find a bounded $u_1 \in \mathbf{L}^G$ such that $\|v - u\|_{\mathbf{L}^G} \leq \varepsilon/2$. Thus

$$\|u - u_1\|_{\mathbf{L}^G} \leq \varepsilon.$$

For every continuous function there exists uniformly convergent sequence of polynomials with rational coefficients. It is easy to check that uniform convergence implies norm convergence in \mathbf{L}^G . This completes the proof. \square

Remark 3.18. It is well known that if G -function does not satisfies Δ_2 condition then \mathbf{L}^G is not separable. One can define a subspace E^G as the closure of bounded functions under Luxemburg norm. In this case, the space E^G is a proper subset of \mathbf{L}^G and is always separable (see [3, 11]).

3.3. Embeddings. We will use the symbols \hookrightarrow nad $\hookrightarrow\hookrightarrow$ for, respectively, continuous and compact embeddings. Recall that

$$F \prec G \iff F(x) \leq G(Kx), |x| \geq M.$$

and

$$F \prec\prec G \iff \lim_{x \rightarrow \infty} \frac{G(\alpha x)}{G(x)} = \infty, \text{ for all } \alpha > 0.$$

Next two theorems provide a basic embeddings for Orlicz spaces.

Proposition 3.19. Assume that $F \prec G$. Then $L^G \hookrightarrow L^F$ and

$$\|u\|_{\mathbf{L}^F} \leq K(C\mu(I) + 1)\|u\|_{\mathbf{L}^G}.$$

for some $C > 0$.

Proof. It is evident that $\mathbf{L}^G \subset \mathbf{L}^F$. Let $u \in \mathbf{L}^G$ and set

$$I_1 = \left\{ t \in I : \left| \frac{u(t)}{K\|u\|_{\mathbf{L}^G}} \right| \leq M \right\}$$

For every $t \in I_1$, we have

$$F\left(\frac{u(t)}{K\|u\|_{\mathbf{L}^G}}\right) \leq G\left(\frac{u(t)}{\|u\|_{\mathbf{L}^G}}\right)$$

and

$$\begin{aligned} \int_I F\left(\frac{u}{K\|u\|_{\mathbf{L}^G}}\right) dt &= \int_{I \setminus I_1} F\left(\frac{u}{K\|u\|_{\mathbf{L}^G}}\right) dt + \int_{I_1} F\left(\frac{u}{K\|u\|_{\mathbf{L}^G}}\right) dt \leq \\ &\leq \mu(I \setminus I_1)\tilde{C} + \int_{I_1} G\left(\frac{u}{\|u\|_{\mathbf{L}^G}}\right) dt \leq \mu(I)\tilde{C} + 1, \end{aligned}$$

where $\tilde{C} = \sup\{G(x) : |x| \leq M\}$. Since $1 \leq \tilde{C}\mu(I) + 1$, we have

$$\int_I F\left(\frac{u(t)}{K(\tilde{C}\mu(I) + 1)\|u\|_{\mathbf{L}^G}}\right) dt \leq 1.$$

Finally,

$$\|u\|_{\mathbf{L}^F} \leq K(\tilde{C}\mu(I) + 1)\|u\|_{\mathbf{L}^G}.$$

□

It is easy to see that there exist constants $C_1, C_2 > 0$ such that $\|u\|_{\mathbf{L}^1} \leq C_1\|u\|_{\mathbf{L}^G}$ and $\|u\|_{\mathbf{L}^G} \leq C_2\|u\|_{\mathbf{L}^\infty}$.

Directly from Lemma 2.4 we obtain that Orlicz spaces can be viewed as a spaces between two Lebesgue spaces determined by constants in Δ_2 and ∇_2 conditions.

Proposition 3.20. For every G there exists $p, q \in (1, \infty)$ such that

$$\mathbf{L}^q \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^p.$$

In particular $\mathbf{L}^\infty \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^1$.

Theorem 3.21 (cf. [6, th. 8.25]). If $F \prec\prec G$ then $\mathbf{L}^G \hookrightarrow\hookrightarrow \mathbf{L}^F$.

Proof. Let $\{u_n\}$ be a bounded sequence in \mathbf{L}^G . Since $\mathbf{L}^G \hookrightarrow \mathbf{L}^F \hookrightarrow\hookrightarrow \mathbf{L}^1$, $\{u_n\}$ is bounded in \mathbf{L}^F and there exists a subsequence, denoted again by $\{u_n\}$, convergent in \mathbf{L}^1 . Hence $\{u_n\}$ converges in measure and thus is Cauchy in measure.

Fix $\varepsilon > 0$ and let $v_{n,k}(t) = (u_n(t) - u_k(t))/\varepsilon$. Since $\{u_n\}$ is bounded in \mathbf{L}^F , there exist $C > 0$ such that $\|u_n\|_{\mathbf{L}^F} \leq C$. There exist $M > 0$ such that if $|x| \geq M$, then

$$F(x) \leq \frac{1}{2}G\left(\frac{x}{C}\right).$$

Set $\overline{F}(M) = \sup\{F(x) : |x| \leq M\}$,

$$I_{n,k} = \left\{t \in I : F(v_{n,k}) \geq \frac{1}{\mu(I)}\right\}, \quad I'_{n,k} = \{t \in I : |v_{n,k}(t)| \geq M\}, \quad I''_{n,k} = I_{n,k} \setminus I'_{n,k}.$$

Since $\{u_n\}$ is Cauchy in measure, there exists N such that if $n, k \geq N$, then $\mu(I''_{n,k}) \leq \mu(I_{n,k}) \leq \frac{1}{2\overline{F}(M)}$. Observe that

- (1) if $t \in I \setminus I_{n,k}$ then $F(v_{n,k}(t)) \leq 1/2\mu(I)$,
- (2) if $t \in I'_{n,k}$, then $F(v_{n,k}(t)) \leq \frac{1}{4}G(v_{n,k}/C)$,
- (3) if $t \in I''_{n,k}$, then $F(v_{n,k}(t)) \leq \overline{F}(M)$.

It follows that for $n, k \geq N$, we have

$$\begin{aligned} \int_I F(v_{n,k}) dt &= \left(\int_{I \setminus I_{n,k}} + \int_{I'_{n,k}} + \int_{I''_{n,k}} \right) F(v_{n,k}) dt \leq \\ &\leq \frac{\mu(I)}{2\mu(I)} + \frac{1}{4} \int_I G\left(\frac{v_{n,k}}{C}\right) dt + \frac{1}{2\overline{F}(M)} \overline{F}(M) \leq 1. \end{aligned}$$

Hence $\|u_n - u_k\|_{\mathbf{L}^F} \leq \varepsilon$ and so $\{u_n\}$ converges in \mathbf{L}^F . \square

In some cases, \mathbf{L}^G is simply a product of $\mathbf{L}^{p_i}(I, \mathbb{R})$, but there exists Orlicz spaces which are not in the form $\mathbf{L}^p(I, \mathbb{R}) \times \mathbf{L}^q(I, \mathbb{R})$ (cf. [9, pp. 18-20]).

Example 3.22. Consider the Orlicz space $\mathbf{L}^G = \mathbf{L}^G(I, \mathbb{R}^2)$ generated, by $G(x) = |x_1|^{p_1} + |x_2|^{p_2}$, $p_1, p_2 > 0$. If $u = (u_1, u_2) \in \mathbf{L}^{p_1}(I, \mathbb{R}) \times \mathbf{L}^{p_2}(I, \mathbb{R})$, then

$$\int_I G(u) dt = \int_I |u_1|^{p_1} dt + \int_I |u_2|^{p_2} dt < \infty.$$

Conversely, if $u = (u_1, u_2) \in \mathbf{L}^G$ then

$$\int_I |u_1|^{p_1} dt \leq \int_I G(u) dt < \infty \text{ and } \int_I |u_2|^{p_2} dt \leq \int_I G(u) dt < \infty.$$

Hence $u \in \mathbf{L}^{p_1}(I, \mathbb{R}) \times \mathbf{L}^{p_2}(I, \mathbb{R})$.

Example 3.23. Consider the Orlicz space $\mathbf{L}^G = \mathbf{L}^G(I, \mathbb{R}^2)$ generated, by $G(x) = (x_1 - x_2)^4 + x_2^2$. From Lemmas 2.4 and 3.20 we obtain that $\mathbf{L}^4(I, \mathbb{R}^2) \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^2(I, \mathbb{R}^2)$. Let u_1 be a function in $\mathbf{L}^2(I, \mathbb{R})$ such that $u_1 \notin \mathbf{L}^p(I, \mathbb{R})$, for $p > 2$. Set $u = (u_1, u_1)$, then

$$\int_I G(u) dt = \int_I |u_1|^2 dt < \infty$$

but

$$\int_I |u|^p dt = \infty.$$

Therefore for every $p > 2$ there exists $u \in \mathbf{L}^G$ such that $u \notin \mathbf{L}^p(I, \mathbb{R}^2)$. Moreover, $u \notin \mathbf{L}^p(I, \mathbb{R}) \times \mathbf{L}^2(I, \mathbb{R})$ for any $p > 2$. From the other hand if $u = (u_1, u_2) \in \mathbf{L}^4(I, \mathbb{R}) \times \mathbf{L}^4(I, \mathbb{R})$ then $u \in \mathbf{L}^G$. Therefore

$$\mathbf{L}^4(I, \mathbb{R}) \times \mathbf{L}^4(I, \mathbb{R}) \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^2(I, \mathbb{R}) \times \mathbf{L}^2(I, \mathbb{R})$$

but \mathbf{L}^G cannot be identified with any

$$\mathbf{L}^4(I, \mathbb{R}) \times \mathbf{L}^4(I, \mathbb{R}) \hookrightarrow \mathbf{L}^p(I, \mathbb{R}) \times \mathbf{L}^q(I, \mathbb{R}) \hookrightarrow \mathbf{L}^2(I, \mathbb{R}) \times \mathbf{L}^2(I, \mathbb{R}).$$

3.4. Duality. Since $\mathbf{L}^G \hookrightarrow \mathbf{L}^p \hookrightarrow \mathbf{L}^{p_0} \hookrightarrow \mathbf{L}^1$ (p given by ∇_2) and $1 < p_0 < p$, it follows that \mathbf{L}^G is closed subspace of reflexive space. Therefore \mathbf{L}^G is reflexive itself.

Theorem 3.24. \mathbf{L}^G is a reflexive Banach space.

The rest of this section is devoted to proving that the general formula for bounded linear operator $F: \mathbf{L}^G \rightarrow \mathbb{R}$ is

$$F(u) = \int_I \langle u, v \rangle dt,$$

where $v \in \mathbf{L}^{G^*}$. We show that the dual space $(\mathbf{L}^G)^*$ can be identified with the Orlicz space \mathbf{L}^{G^*} generated by conjugate function G^* . On the other hand, $(G^*)^* = G$ and $(\mathbf{L}^G)^* \simeq \mathbf{L}^{G^*}$ implies reflexivity as well.

Lemma 3.25. Every $v \in \mathbf{L}^{G^*}$ can be identified with the following functional $F_v \in (\mathbf{L}^G)^*$:

$$F_v(u) = \int_I \langle u, v \rangle dt.$$

Moreover $\|F_v\| \leq 2\|v\|_{\mathbf{L}^{G^*}}$.

Proof. It is easy to see that F_v is linear. By the Hölder inequality we get

$$F_v(u) = \int_I \langle u, v \rangle dt \leq 2\|u\|_{\mathbf{L}^G} \|v\|_{\mathbf{L}^{G^*}}.$$

Thus F_v is bounded and $\|F_v\| \leq 2\|v\|_{\mathbf{L}^{G^*}}$. □

Lemma 3.26 (cf. [10, 11]). If $v \in \mathbf{L}^1(I, \mathbb{R}^n)$ is such that for each piecewise constant function $u \in \mathbf{L}^G$ satisfy

$$\int_I \langle u, v \rangle dt \leq M\|u\|_{\mathbf{L}^G},$$

then $v \in \mathbf{L}^{G^*}$ and $\|v\|_{\mathbf{L}^{G^*}} \leq M$.

Proof. Define an approximation

$$v_{n,i} = \frac{n}{\mu(I)} \int_{E_i} v dt, \quad E_i \text{ - disjoint, } \quad I = \bigcup_{i=1}^n E_i, \quad \mu(E_i) = \frac{\mu(I)}{n}.$$

Set $v_n = \sum_{i=1}^n v_{n,i} \chi_{E_i}$. Let $u \in \mathbf{L}^G$ be a simple function, define approximation u_n of u in the same way. By Jensen inequality

$$\begin{aligned} \int_I G\left(\frac{u_n}{\|u\|_{\mathbf{L}^G}}\right) dt &= \sum_{i=1}^n \mu(E_i) G\left(\frac{1}{\mu(E_i)} \int_{E_i} \frac{u}{\|u\|_{\mathbf{L}^G}} dt\right) \leq \\ &\leq \sum_{i=1}^n \mu(E_i) \frac{1}{\mu(E_i)} \int_{E_i} G\left(\frac{u}{\|u\|_{\mathbf{L}^G}}\right) dt = \int_I G\left(\frac{u}{\|u\|_{\mathbf{L}^G}}\right) dt = 1. \end{aligned}$$

Hence $\|u_n\|_{\mathbf{L}^G} \leq \|u\|_{\mathbf{L}^G}$. A direct computation yields

$$\int_I \langle u, v_n \rangle dt = \int_I \langle u_n, v \rangle dt \leq M\|u_n\|_{\mathbf{L}^G} \leq M\|u\|_{\mathbf{L}^G}.$$

We can find for each $v_{n,i}$ a $z_{n,i} \in \mathbb{R}^n$ such that $\langle z_{n,i}, v_{n,i}/M \rangle = G(z_{n,i}) + G^*(v_{n,i}/M)$. Suppose that $\sum_{i=1}^n \mu(E_i) G(z_{n,i}) > 1$. Then there exists $\beta < 1$ such that $\sum_{i=1}^n \mu(E_i) G(\beta z_{n,i}) = 1$. Putting

$$u = \sum_{i=1}^n \beta z_{n,i} \chi_{E_i}$$

we obtain that $\int_I G(u) dt \leq 1$ and $\|u\|_{\mathbf{L}^G} \leq 1$. Therefore

$$\begin{aligned} \sum_{i=1}^n \mu(E_i) G^*(v_{n,i}/M) &= \frac{1}{M} \sum_{i=1}^n \mu(E_i) \langle z_{n,i}, v_{n,i} \rangle - \sum_{i=1}^n \mu(E_i) G(z_{n,i}) = \\ &= \frac{1}{M\beta} \int_I \langle u, v_n \rangle dt - \sum_{i=1}^n \mu(E_i) G(z_{n,i}) \leq \frac{1}{\beta} - \frac{1}{\beta} \sum_{i=1}^n \mu(E_i) G(\beta z_{n,i}) \leq 0. \end{aligned}$$

Now assume that $\mu(E_i) \sum G(z_{n,i}) \leq 1$ and repeat the same computation with $\beta = 1$ and obtain

$$\sum_{i=1}^n \mu(E_i) G^*(v_{n,i}/M) \leq 1.$$

In both cases we get

$$\int_I G^*(v_n/M) dt \leq 1.$$

Since $v_n \rightarrow v$ a.e. we can conclude that $G^*(v/M) \leq \lim G^*(v_n/M)$. By the Fatou theorem we get

$$\int_I G^*(v/M) dt \leq 1.$$

□

Lemma 3.27 (cf. [3, 11]). For every $F \in (\mathbf{L}^G)^*$ there exists unique $v \in \mathbf{L}^{G^*}$ such that for every $u \in \mathbf{L}^G$

$$Fu = \int_I \langle u, v \rangle dt.$$

Proof. For a measurable subset $E \subset I$ define $\chi_E^N(x) = (\chi_E, \dots, \chi_E)$. Note that $\chi_E^N \in \mathbf{L}^G$. Set

$$\phi(E) = F(\chi_E^N).$$

For every sequence $\{E_i\}$ of measurable and pairwise disjoint subsets of I such that $E = \bigcup E_i$ we have $\chi_E^N = \sum \chi_{E_i}^N$ and

$$\phi(E) = F(\chi_E^N) = F\left(\sum \chi_{E_i}^N\right) = \sum F(\chi_{E_i}^N) = \sum \phi(E_i).$$

Suppose that there exists a sequence $\{E_i\}$ of measurable sets and $\delta > 0$ such that $\mu(E_i) \rightarrow 0$ and $\|\chi_{E_i}^N\|_{\mathbf{L}^G} > \delta$ for all i . Then

$$1 < \int_I G\left(\frac{\chi_{E_i}^N}{\delta}\right) dt = \int_{E_i} G\left(\frac{(1, \dots, 1)}{\delta}\right) dt = \mu(E_i) G\left(\frac{(1, \dots, 1)}{\delta}\right).$$

A contradiction. From inequality

$$|\phi(E_i)| \leq \|F\| \|\chi_{E_i}^N\|_{\mathbf{L}^G}$$

we obtain that if $\mu(E_i) \rightarrow 0$ then $|\phi(E_i)| \rightarrow 0$. Thus a set function ϕ is σ -additive and absolutely continuous with respect to Lebesgue measure.

It follows from the Radon-Nikodym theorem that there exists a function $v \in \mathbf{L}^1(I, \mathbb{R}^N)$ such that

$$F(\chi_E^N) = \phi(E) = \int_I \langle \chi_E^N, v \rangle dt.$$

For every step function $u = \sum c_i \chi_{E_i}$, by linearity of F ,

$$F(u) = F\left(\sum c_i \chi_{E_i}\right) = \sum c_i F(\chi_{E_i}) = \sum c_i \int_I \langle \chi_{E_i}, v \rangle dt = \int_I \langle u, v \rangle dt.$$

By lemma 3.26 we get that $v \in \mathbf{L}^{G^*}$. Assume now that u is bounded. Choose a sequence of step functions $\{u_n\}$ such that

$$u_n(t) = \sum c_i \chi_{E_i}, \quad c_i = \frac{1}{\mu(E_i)} \int_{E_i} u dt,$$

where E_i are disjoint and

$$\mu(E_i) = \frac{\mu(I)}{n}, \quad I = \bigcup_{i=1}^n E_i.$$

Clearly, $u_n \rightarrow u$ a.e. and the sequence $\{u_n\}$ is uniformly bounded. It follows that

$$F(u) = \lim_{n \rightarrow \infty} F(u_n) = \lim_{n \rightarrow \infty} \int_I \langle u_n, v \rangle dt = \int_I \langle u, v \rangle dt.$$

Suppose that u is an arbitrary function in \mathbf{L}^G . There exists a sequence $\{u_n\}$ of bounded functions which converges a.e. to u such that $|u_n(t)| \leq |u(t)|$ a.e. Thus

$$F(u) = \lim_{n \rightarrow \infty} F(u_n) = \lim_{n \rightarrow \infty} \int_I \langle u_n, v \rangle dt = \int_I \langle u, v \rangle dt.$$

It remains to show that v is unique. Suppose that v_1 and v_2 represent F . Then we have

$$\int_I \langle u, v_1 \rangle dt = \int_I \langle u, v_2 \rangle dt$$

for all $u \in \mathbf{L}^\infty$. Thus $v_1 = v_2$. □

As a consequence we obtain that $\mathbf{L}^{G^*} \simeq (\mathbf{L}^G)^*$. Since $G^{**} = G$, we also get $\mathbf{L}^G \simeq (\mathbf{L}^{G^*})^*$.

Remark 3.28. If G -function does not satisfies Δ_2 condition then \mathbf{L}^G is not reflexive and $(\mathbf{L}^G)^*$ is not isomorphic to \mathbf{L}^{G^*} (see [3, 11]).

4. ORLICZ-SOBOLEV SPACES

The Orlicz-Sobolev space $\mathbf{W}^1 \mathbf{L}^G = \mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^n)$ is defined to be

$$\mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^n) := \{u \in \mathbf{L}^G(I, \mathbb{R}^n) : \dot{u} \in \mathbf{L}^G(I, \mathbb{R}^n)\}.$$

For $u \in \mathbf{W}^1 \mathbf{L}^G$ we define

$$\|u\|_{\mathbf{W}^1 \mathbf{L}^G} := \|u\|_{\mathbf{L}^G} + \|\dot{u}\|_{\mathbf{L}^G}.$$

Define $\mathbf{W}_0^1 \mathbf{L}^G = \mathbf{W}_0^1 \mathbf{L}^G(I, \mathbb{R}^n)$ as the closure of $C_0^1(I, \mathbb{R}^n)$ in $\mathbf{W}^1 \mathbf{L}^G$ with respect to the $\|\cdot\|_{\mathbf{W}^1 \mathbf{L}^G}$.

Theorem 4.1. The space $(\mathbf{W}^1 \mathbf{L}^G, \|\cdot\|_{\mathbf{W}^1 \mathbf{L}^G})$ is a separable reflexive Banach space.

Proof is standard and will be omitted, see for instance [26]. If $G(x) = \frac{1}{p}|x|^p$, then the Orlicz-Sobolev space $\mathbf{W}^1 \mathbf{L}^G$ coincides with the Sobolev space $\mathbf{W}^{1,p}(I, \mathbb{R}^n)$. Observe that $u_n \rightarrow u$ in $\mathbf{W}^1 \mathbf{L}^G$ is equivalent to $R_G(u_n - u) \rightarrow 0$ and $R_G(\dot{u}_n - \dot{u}) \rightarrow 0$.

On $\mathbf{W}^1 \mathbf{L}^G$ one can introduce another norm (cf. [27]):

$$\|u\|_{1, \mathbf{W}^1 \mathbf{L}^G} = \inf\{\alpha > 0 : \int_I G\left(\frac{u}{\alpha}\right) + G\left(\frac{\dot{u}}{\alpha}\right) dt \leq 1\}.$$

Proposition 4.2. A function $\|\cdot\|_{1, \mathbf{W}^1 \mathbf{L}^G}$ is an equivalent norm on $\mathbf{W}^1 \mathbf{L}^G$. Moreover

$$\|u\|_{\mathbf{W}^1 \mathbf{L}^G} \leq 2\|u\|_{1, \mathbf{W}^1 \mathbf{L}^G} \leq 4\|u\|_{\mathbf{W}^1 \mathbf{L}^G}.$$

Proof. The proof that $\|\cdot\|_{1, \mathbf{W}^1 \mathbf{L}^G}$ is a norm is similar to the proof of Theorem 3.2 and is left to the reader. For the other part, note that

$$\int_I G\left(\frac{u}{\|u\|_{1, \mathbf{W}^1 \mathbf{L}^G}}\right) + G\left(\frac{\dot{u}}{\|u\|_{1, \mathbf{W}^1 \mathbf{L}^G}}\right) dt \leq 1$$

implies

$$\int_I G\left(\frac{u}{\|u\|_{1, \mathbf{W}^1 \mathbf{L}^G}}\right) dt \leq 1 \text{ and } \int_I G\left(\frac{\dot{u}}{\|u\|_{1, \mathbf{W}^1 \mathbf{L}^G}}\right) dt \leq 1.$$

From this $\|u\|_{\mathbf{L}^G} \leq \|u\|_{1, \mathbf{W}^1 \mathbf{L}^G}$ and $\|\dot{u}\|_{\mathbf{L}^G} \leq \|\dot{u}\|_{1, \mathbf{W}^1 \mathbf{L}^G}$ and finally, $\|u\|_{\mathbf{W}^1 \mathbf{L}^G} \leq 2\|u\|_{1, \mathbf{W}^1 \mathbf{L}^G}$. Let $\alpha = \max\{\|u\|_{\mathbf{L}^G}, \|\dot{u}\|_{\mathbf{L}^G}\}$. Since $\|u\|_{\mathbf{L}^G}, \|\dot{u}\|_{\mathbf{L}^G} \leq \alpha$,

$$G\left(\frac{u(t)}{\alpha}\right) \leq \frac{\|u\|_{\mathbf{L}^G}}{\alpha} G\left(\frac{u(t)}{\|u\|_{\mathbf{L}^G}}\right)$$

and

$$G\left(\frac{u(t)}{\alpha}\right) \leq \frac{\|\dot{u}\|_{\mathbf{L}^G}}{\alpha} G\left(\frac{u(t)}{\|\dot{u}\|_{\mathbf{L}^G}}\right).$$

Using the above relations, we obtain

$$\begin{aligned} \int_I G\left(\frac{u}{2\alpha}\right) + G\left(\frac{\dot{u}}{2\alpha}\right) dt &\leq \frac{1}{2} \int_I G\left(\frac{u}{\alpha}\right) + G\left(\frac{\dot{u}}{\alpha}\right) dt \leq \\ &\leq \frac{1}{2} \frac{\|u\|_{\mathbf{L}^G}}{\alpha} \int_I G\left(\frac{u}{\alpha}\right) dt + \frac{1}{2} \frac{\|\dot{u}\|_{\mathbf{L}^G}}{\alpha} \int_I G\left(\frac{\dot{u}}{\alpha}\right) dt \leq 1 \end{aligned}$$

This implies $\|u\|_{1, \mathbf{W}^1 \mathbf{L}^G} \leq 2\alpha \leq 2\|u\|_{\mathbf{W}^1 \mathbf{L}^G}$ □

Since there exist $p, q \in (1, \infty)$ such that $\mathbf{L}^q \hookrightarrow \mathbf{L}^G \hookrightarrow \mathbf{L}^p$, the following continuous embeddings exist

$$\mathbf{W}^{1,q} \hookrightarrow \mathbf{W}^1 \mathbf{L}^G \hookrightarrow \mathbf{W}^{1,p}$$

Using standard results from the theory of Sobolev spaces we get

- (1) $\mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^n) \hookrightarrow \mathbf{W}^{1,1}$
- (2) $\mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^n) \hookrightarrow \mathbf{L}^q$, for all $1 \leq q \leq \infty$
- (3) $\mathbf{W}^1 \mathbf{L}^G(I, \mathbb{R}^n) \hookrightarrow C(\bar{I})$

As a consequence we have

Theorem 4.3. A function $u \in W^1 L^G$ is absolutely continuous. Precisely, there exist absolutely continuous representative of u such that for all $a, b \in I$

$$u(b) - u(a) = \int_a^b \dot{u}(t) dt.$$

Directly from definition of $\mathbf{W}_0^1 \mathbf{L}^G$ we obtain important property of functions in $\mathbf{W}_0^1 \mathbf{L}^G$.

Theorem 4.4. If $u \in \mathbf{W}_0^1 \mathbf{L}^G$ then $u = 0$ on ∂I .

Using embeddings mentioned above we have for every $u \in \mathbf{W}^1 \mathbf{L}^G$

$$(3) \quad \|u\|_{\mathbf{L}^\infty} \leq C \|u\|_{\mathbf{W}^1 \mathbf{L}^G}.$$

Theorem 4.5 (Sobolev inequality). For every function $u \in \mathbf{W}^1 \mathbf{L}^G$

$$\|u - u_I\|_{\mathbf{L}^G} \leq \mu(I) \|\dot{u}\|_{\mathbf{L}^G}$$

where $u_I = \frac{1}{\mu(I)} \int_I u$.

Proof. Since u is absolutely continuous, there exists $t_0 \in I$ such that $u(t_0) = \frac{1}{\mu(I)} \int_I u$ and for every $t \in I$ we have

$$u(t) - u(t_0) = \int_{t_0}^t \dot{u} dt.$$

By Jensen's inequality,

$$\begin{aligned} G\left(\frac{u(t) - u(t_0)}{\mu(I) \|\dot{u}\|_{\mathbf{L}^G}}\right) &= G\left(\frac{1}{|t - t_0|} \int_{t_0}^t \frac{\dot{u}}{\mu(I)} dt\right) \leq \\ &\leq \frac{1}{|t - t_0|} \int_{t_0}^t G\left(\frac{|t - t_0|}{\mu(I)} \frac{\dot{u}}{\|\dot{u}\|_{\mathbf{L}^G}}\right) dt \leq \frac{1}{\mu(I)} \int_I G\left(\frac{\dot{u}}{\|\dot{u}\|_{\mathbf{L}^G}}\right) dt \leq \frac{1}{\mu(I)}. \end{aligned}$$

Integrating both sides over I we get

$$\int_I G\left(\frac{u - u(t_0)}{\mu(I)\|\dot{u}\|_{\mathbf{L}^G}}\right) dt \leq 1.$$

Thus $\|u - u_I\|_{\mathbf{L}^G} \leq \mu(I)\|\dot{u}\|_{\mathbf{L}^G}$ □

In similar way we get

Theorem 4.6 (Poincare inequality). For every $u \in \mathbf{W}_0^1 \mathbf{L}^G$

$$\|u\|_{\mathbf{L}^G} \leq \mu(I)\|\dot{u}\|_{\mathbf{L}^G}.$$

It follows that one can introduce equivalent norm in $\mathbf{W}_0^1 \mathbf{L}^G$:

$$\|u\|_{\mathbf{W}_0^1 \mathbf{L}^G} = \|\dot{u}\|_{\mathbf{L}^G}.$$

Every linear functional F on $\mathbf{W}_0^1 \mathbf{L}^G$ can be represented in the form

$$F(u) = \int_I \langle u, v_0 \rangle + \langle \dot{u}, v_1 \rangle dt.$$

Where $v_0, v_1 \in \mathbf{L}^{G^*}$. Moreover,

$$\|F\| = \max\{\|v_0\|_{\mathbf{L}^{G^*}}, \|v_1\|_{\mathbf{L}^{G^*}}\}.$$

In the case of Sobolev space $\mathbf{W}^{1,p}$ the proof is given in [26, proposition 8.14], but it remains the same for Orlicz-Sobolev spaces. As was pointed out in [26], the first assertion of the above proposition holds for every linear functional on $\mathbf{W}^1 \mathbf{L}^G$.

5. VARIATIONAL SETTING

In this section we examine the principal part

$$(4) \quad \mathcal{I}(u) = \int_I F(t, u, \dot{u}) dt$$

of the variational functional associated with Euler-Lagrange equation

$$\frac{d}{dt} F_v(t, u, \dot{u}) = F_x(t, u, \dot{u}) + \nabla V(t, u), \quad t \in I$$

where $u: I \rightarrow \mathbb{R}^N$ and the Lagrangian $L: I \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ is given by $L(t, x, v) = F(t, x, v) + V(t, x)$.

In definition of the Orlicz space we need not to assume that G is differentiable, but when we consider the functional \mathcal{I} we need it to show that $\mathcal{I} \in C^1$. Throughout this section we will assume, in addition to (G_1) – (G_6) , that G satisfies

(G_7) G is of a class C^1 .

Remark 5.1. Differentiability of f is not sufficient to differentiability of f^* . But if f is finite, strictly convex, 1-coercive and differentiable then so is f^* . This result is in close relation with Legendre duality (see [21, p. 239] and [1] for more details).

It is well known that if G is continuously differentiable then for all $x, y \in \mathbb{R}^n$

$$(5) \quad G(x) - G(x - y) \leq \langle \nabla G(x), y \rangle \leq G(x + y) - G(x)$$

and

$$\langle x, \nabla G(x) \rangle = G(x) + G^*(\nabla G(x)).$$

Let $y = x$ in (5). Then $\langle \nabla G(x), x \rangle \leq G(2x) - G(x)$. Therefore, for all $x \in \mathbb{R}^N$

$$G^*(\nabla G(x)) \leq G(2x).$$

Directly from the above we get

Proposition 5.2. If $u \in \mathbf{L}^G$ then $\nabla G(u) \in \mathbf{L}^{G^*}$.

Lemma 5.3 (cf. [16, lemma A.5]). If $u_n \rightarrow u$ in \mathbf{L}^G then $R_{G^*}(\nabla G(u_n)) \rightarrow R_{G^*}(\nabla G(u))$.

Proof. There exists a subsequence $\{u_{n_k}\}$ such that $u_{n_k} \rightarrow u$ a.e., $G(u_{n_k}) \rightarrow G(u)$ a.e. and $G(u_{n_k}) \leq h \in \mathbf{L}^1(I, \mathbb{R})$. By continuity of ∇G and G^* we have $\nabla G(u_{n_k}) \rightarrow \nabla G(u)$ a.e. and

$$G^*(\nabla G(u_{n_k})) \rightarrow G^*(\nabla G(u)) \text{ a.e.}$$

Since $G^*(\nabla G(x)) \leq G(2x)$,

$$G^*(\nabla G(u_{n_k})) \leq G(2u_{n_k}) \leq C + K_1 G(u_{n_k}) \leq C + K_1 h.$$

By dominated convergence theorem $R_{G^*}(\nabla G(u_{n_k})) \rightarrow R_{G^*}(\nabla G(u))$. Since this holds for any subsequence of $\{u_n\}$ we have that

$$R_{G^*}(\nabla G(u_n)) \rightarrow R_{G^*}(\nabla G(u)).$$

□

As a direct consequence of the above lemma and Lemma 3.13 we obtain

Proposition 5.4.

$$\|u_n - u\|_{\mathbf{L}^G} \rightarrow 0 \implies \|\nabla G(u_n) - \nabla G(u)\|_{\mathbf{L}^{G^*}} \rightarrow 0.$$

5.1. **Case I.** We shall first examine a special case $F(t, x, v) = G(v)$, now functional (4) takes the form

$$\mathcal{I}(u) = \int_I G(\dot{u}) dt.$$

Theorem 5.5. $\mathcal{I} \in C^1(\mathbf{W}^1 \mathbf{L}^G, \mathbb{R})$. Moreover

$$(6) \quad \mathcal{I}'(u)v = \int_I \langle \nabla G(\dot{u}), \dot{\varphi} \rangle dt.$$

Proof. The proof follows similar lines as [2, th. 3.2] (see also [1, thm 1.4]). First, note that $\dot{u} \in \mathbf{L}^G$ implies

$$0 \leq \mathcal{I}(u) < \infty.$$

It suffices to show that \mathcal{I} has at every point u directional derivative $\mathcal{I}'(u) \in (\mathbf{W}^1 \mathbf{L}^G)^*$ given by (6) and that the mapping $\mathcal{I}' : \mathbf{W}^1 \mathbf{L}^G \rightarrow (\mathbf{W}^1 \mathbf{L}^G)^*$, is continuous.

Let $u \in \mathbf{W}^1 \mathbf{L}^G$, $\varphi \in \mathbf{W}^1 \mathbf{L}^G \setminus \{0\}$, $t \in I$, $s \in [-1, 1]$. Define

$$H(s, t) := G(\dot{u}(t) + s\dot{\varphi}(t)).$$

By (5) we obtain

$$\int_I |H_s(s, t)| dt = \int_I |\langle \nabla G(\dot{u} + s\dot{\varphi}), \dot{\varphi} \rangle| dt \leq \int_I G(\dot{u} + (s+1)\dot{\varphi}) + \int_I G(\dot{u} + s\dot{\varphi}) dt < \infty.$$

Consequently, \mathcal{I} has a directional derivative and

$$\mathcal{I}'(u)\varphi = \frac{d}{ds} \mathcal{I}(u + s\varphi) \Big|_{s=0} = \int_I \langle \nabla G(\dot{u}), \dot{\varphi} \rangle dt.$$

By Lemma 5.2 and Hölder inequality

$$|\mathcal{I}'(u)\varphi| = \left| \int_I \langle \nabla G(\dot{u}), \dot{\varphi} \rangle dt \right| \leq 2 \|\nabla G(\dot{u})\|_{\mathbf{L}^{G^*}} \|\varphi\|_{\mathbf{L}^G} \leq C \|\varphi\|_{\mathbf{W}^1 \mathbf{L}^G}.$$

To finish the proof it suffices to show that if $u_n \rightarrow u$ in $\mathbf{W}^1 \mathbf{L}^G$, then $\mathcal{I}'(u_n) \rightarrow \mathcal{I}'(u)$ in $(W^1 L^G)^*$. Using Hölder inequality and Proposition 5.4 we obtain

$$|\mathcal{I}'(u_n)\varphi - \mathcal{I}'(u)\varphi| = \left| \int_I \langle \nabla G(\dot{u}_n) - \nabla G(\dot{u}), \dot{\varphi} \rangle dt \right| \leq 2 \|\nabla G(\dot{u}_n) - \nabla G(\dot{u})\|_{\mathbf{L}^{G^*}} \|\dot{\varphi}\|_{\mathbf{L}^G} \rightarrow 0.$$

□

5.2. Case II. We turn to general case. Suppose that $F: I \times \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies

- (F₁) $F \in C^1$
- (F₂) $|F(t, x, v)| \leq a(|x|)(b(t) + G(v)),$
- (F₃) $|F_x(t, x, v)| \leq a(|x|)(b(t) + G(v)),$
- (F₄) $G^*(F_v(t, x, v)) \leq a(|x|)(c(t) + G^*(\nabla G(v))).$

where $a \in C(\mathbb{R}_+, \mathbb{R}_+)$, $b, c \in \mathbf{L}^1(I, \mathbb{R}_+)$.

If $G(v) = |v|^p$ then conditions (F₂), (F₃) and (F₄) take the standard form (Theorem 1.4 from [1]). In [2] there are similar conditions with $G(v) = \Phi(|v|)$, where Φ is an N-function. In this case, condition (F₄) takes the form $|F_v(t, x, v)| \leq \tilde{a}(|x|)(\tilde{c}(t) + \Phi'(|u|))$. In anisotropic case we need to use G^* , because vector valued G-function is not necessarily monotone with respect to $|\cdot|$.

Directly from (F₃), (F₄) and Proposition 5.2 we have

Lemma 5.6. If $u \in \mathbf{W}^1 \mathbf{L}^G$, then $F_x(\cdot, u, \dot{u}) \in \mathbf{L}^1$ and $F_v(\cdot, u, \dot{u}) \in \mathbf{L}^{G^*}$.

Proof. Define non decreasing function

$$\alpha(s) = \sup_{\tau \in [0, s]} a(\tau).$$

Then, for $u \in \mathbf{W}^1 \mathbf{L}^G$ we have

$$(7) \quad a(|u(t)|) \leq \alpha(\|u\|_{\mathbf{L}^\infty}) \leq \alpha(C\|u\|_{\mathbf{W}^1 \mathbf{L}^G}).$$

Let $u \in \mathbf{W}^1 \mathbf{L}^G$. By (7) and (F₃)

$$\int_I |F_x(t, u, \dot{u})| dt \leq \int_I a(|u(t)|)(b(t) + G(\dot{u})) dt \leq \alpha(C\|u\|_{\mathbf{W}^1 \mathbf{L}^G}) \int_I (b(t) + G(\dot{u})) dt < \infty.$$

Moreover, by Proposition 5.2 and (F₄)

$$\int_I G^*(F_v(t, u, \dot{u})) dt \leq \alpha(C\|u\|_{\mathbf{W}^1 \mathbf{L}^G}) \int_I (c(t) + G^*(\nabla G(\dot{u}))) dt < \infty.$$

□

Theorem 5.7. $\mathcal{I} \in C^1(\mathbf{W}^1 \mathbf{L}^G, \mathbb{R})$. Moreover

$$(8) \quad \mathcal{I}'(u)\varphi = \int_I \langle F_x(t, u, \dot{u}), \varphi \rangle dt + \int_I \langle F_v(t, u, \dot{u}), \dot{\varphi} \rangle dt.$$

Proof. By (F₂),

$$|\mathcal{I}(u)| \leq \int_I a(|u|)(b(t) + G(\dot{u})) dt \leq \alpha(\|u\|_{\mathbf{W}^1 \mathbf{L}^G}) \int_I (b(t) + G(\dot{u})) dt < \infty.$$

It suffices to show that directional derivative $\mathcal{I}'(u) \in (\mathbf{W}^1 \mathbf{L}^G)^*$ exists, is given by (8) and that the mapping $\mathcal{I}': \mathbf{W}^1 \mathbf{L}^G \rightarrow (\mathbf{W}^1 \mathbf{L}^G)^*$ is continuous.

Let $u \in \mathbf{W}^1 \mathbf{L}^G$, $\varphi \in \mathbf{W}^1 \mathbf{L}^G \setminus \{0\}$, $t \in I$, $s \in [-1, 1]$. Define

$$H(s, t) := F(t, u + s\varphi, \dot{u} + s\dot{\varphi}).$$

By (F₃), continuity of φ , (7) and the fact that $u + s\varphi \in \mathbf{W}^1 \mathbf{L}^G$ we obtain

$$\begin{aligned} \int_I |\langle F_x(t, u + s\varphi, \dot{u} + s\dot{\varphi}), \varphi \rangle| dt &\leq \int_I |F_x(t, u + s\varphi, \dot{u} + s\dot{\varphi})| |\varphi| dt \leq \\ &\leq \int_I a(|u + s\varphi|)(b(t) + G(\dot{u} + s\dot{\varphi})) |\varphi| dt \leq \\ &\leq \alpha(\|u + s\varphi\|_{\mathbf{W}^1 \mathbf{L}^G}) \int_I (b(t) + G(\dot{u} + s\dot{\varphi})) |\varphi| dt < \infty. \end{aligned}$$

By the Fenchel inequality, (F₄) and Lemma 5.6 we obtain

$$\int_I |\langle F_v(t, u + s\varphi, \dot{u} + s\dot{\varphi}), \dot{\varphi} \rangle| dt \leq \int_I [G^*(F_v(t, u + s\varphi, \dot{u} + s\dot{\varphi})) + G(\dot{\varphi})] dt < \infty.$$

It follows that

$$\int_I |H_s(s, t)| dt = \int_I |\langle F_x(t, u + s\varphi, \dot{u} + s\dot{\varphi}), \varphi \rangle + \langle F_v(t, u + s\varphi, \dot{u} + s\dot{\varphi}), \varphi \rangle| dt < \infty.$$

Consequently, \mathcal{I} has a directional derivative and

$$\mathcal{I}'(u)\varphi = \frac{d}{ds} \mathcal{I}(u + s\varphi) \Big|_{s=0} = \int_I \langle F_x(t, u, \dot{u}), \varphi \rangle dt + \int_I \langle F_v(t, u, \dot{u}), \dot{\varphi} \rangle dt.$$

By Lemma 5.6, the Hölder inequality and (3) we get

$$|\mathcal{I}'(u)\varphi| \leq \|F_x(t, u, \dot{u})\|_{\mathbf{L}^1} \|\varphi\|_{\mathbf{L}^\infty} + \|F_v(t, u, \dot{u})\|_{\mathbf{L}^{G^*}} \|\dot{\varphi}\|_{\mathbf{L}^G} \leq C \|\varphi\|_{\mathbf{W}^1 \mathbf{L}^G}.$$

To finish the proof it suffices to show that \mathcal{I}' is continuous. Since $u_n \rightarrow u$ in $\mathbf{W}^1 \mathbf{L}^G$, it follows that $u_n \rightarrow u$ in \mathbf{L}^G , $\dot{u}_n \rightarrow \dot{u}$ in \mathbf{L}^G and there exists $M > 0$ such that $\|u_n\|_{\mathbf{W}^1 \mathbf{L}^G} < M$.

By Lemma 3.12 we have $G(\dot{u}_n) \rightarrow G(\dot{u})$ in $\mathbf{L}^1(I, \mathbb{R})$. Hence there exists a subsequence $\{u_{n_k}\}$ and $h \in \mathbf{L}^1(I, \mathbb{R})$ such that

$$G(\dot{u}_{n_k}) \rightarrow G(\dot{u}) \text{ a.e and } G(\dot{u}_{n_k}) \leq h.$$

By (F_3) and since $\{u_{n_k}\}$ is bounded, we obtain

$$|F_x(t, u_{n_k}, \dot{u}_{n_k})| \leq \alpha(\|u_{n_k}\|_{\mathbf{W}^1 \mathbf{L}^G})(b(t) + G(\dot{u}_{n_k})) dt \leq \alpha(M)(b(t) + h(t)).$$

By (F_1) we have

$$F_x(t, u_{n_k}, \dot{u}_{n_k}) \rightarrow F_x(t, u, \dot{u})$$

for a.e $t \in I$. Applying Lebesgue Dominated Convergence Theorem we obtain

$$\int_I \langle F_x(t, u_{n_k}, \dot{u}_{n_k}), \varphi \rangle dt \rightarrow \int_I \langle F_x(t, u, \dot{u}), \varphi \rangle dt.$$

Since this holds for any subsequence of $\{u_n\}$ we have that

$$\int_I \langle F_x(t, u_n, \dot{u}_n), \varphi \rangle dt \rightarrow \int_I \langle F_x(t, u, \dot{u}), \varphi \rangle dt.$$

By (F_4) and Lemma 5.6

$$G^*(F_v(t, u_{n_k}, \dot{u}_{n_k})) \leq \alpha(M)(c(t) + G^*(\nabla G(\dot{u}_{n_k}))).$$

In the same way as in the proof of Lemma 5.3 we obtain

$$G^*(F_v(t, u_{n_k}, \dot{u}_{n_k})) \leq \alpha(M)(c(t) + C + K_1 h(t)).$$

By continuity of F_v we obtain

$$G^*(F_v(t, u_{n_k}, \dot{u}_{n_k})) \rightarrow G^*(F_v(t, u, \dot{u}))$$

for a.e $t \in I$ and consequently

$$\int_I G^*(F_v(t, u_{n_k}, \dot{u}_{n_k})) dt \rightarrow \int_I G^*(F_v(t, u, \dot{u})) dt.$$

It follows that

$$\int_I G^*(F_v(t, u_n, \dot{u}_n)) dt \rightarrow \int_I G^*(F_v(t, u, \dot{u})) dt.$$

Application of Lemma 3.13 to R_{G^*} yields $\|F_v(\cdot, u_n, \dot{u}_n) - F_v(\cdot, u, \dot{u})\|_{\mathbf{L}^{G^*}} \rightarrow 0$. By Hölder inequality

$$\left| \int_I \langle F_v(t, u_n, \dot{u}_n) - F_v(t, u, \dot{u}), \dot{\varphi} \rangle dt \right| \leq 2 \|F_v(\cdot, u_n, \dot{u}_n) - F_v(\cdot, u, \dot{u})\|_{\mathbf{L}^{G^*}} \|\dot{\varphi}\|_{\mathbf{L}^G} \rightarrow 0.$$

Finally,

$$\int_I \langle F_v(t, u_n, \dot{u}_n), \dot{\varphi} \rangle dt \rightarrow \int_I \langle F_v(t, u, \dot{u}), \dot{\varphi} \rangle dt.$$

□

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